

# Exam Computational Methods of Science

June 18th 2026 08:30-11:30 (2 hours)

This exam is 'closed book'. It is NOT allowed to use a book, lecture notes or electronic devices.

**Always give a clear explanation of your answer.** An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Free points: 10

$$q = \frac{du}{dx} + au$$

## Part 1

1. Consider the o.d.e.  $\frac{d^2u}{dx^2} + a \frac{du}{dx} = bx(2-x)$  on the domain  $[0, 2]$ .

The parameters  $(a, b)$  and the boundary conditions will be determined below.

- ~~1a~~ 10 The term  $\frac{d^2u}{dx^2}$  is treated with 2nd order central discretisation (FDM).

The term  $\frac{du}{dx}$  is treated with 1st order backward ('to-the-left') discretisation (FDM).

An equidistant grid is used, where the boundary points match  $x=0$  and  $x=2$ .

This gives  $A\vec{u} = \vec{b}$ , where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 & 0 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

~~1b~~ Determine the values of  $a$  and  $b$ .

~~2~~ What are the bound. cond.? What is the accuracy of their implementation?

- (b) 10 Let  $(a, b) = (-2, 2)$ .

The o.d.e. can be written in conservation form:  $\frac{\partial q}{\partial x} = f(x)$ , which makes FVM (Finite Volume Discretisation) an appropriate technique.

The gridpoints on the equidistant grid are numbered  $i = 0 \dots 4$ .

The boundaries  $x=0$  and  $x=1$  are located halfway gridpoints, at  $i = \frac{1}{2}$  and  $i = 3\frac{1}{2}$ .

(1) Apply FVM discretisation, and determine the row of the linear system (i.e. values of the entries of the matrix and RHS vector) that is related to location  $x=1$ .

(2) Give the FVM implementation of the boundary conditions obtained in (a).

- ~~2~~ 10 Use Taylor series to construct a FDM discretisation for  $\frac{\partial^2 u}{\partial x^2}$  on a stretched(!) grid (see figure) and show that the truncation error  $\tau = O(h_2 - h_1)$ .

$$\begin{array}{ccccccc} & h_1 & & h_2 & & & \\ | & & | & & | & & | \\ x_{i-1} & & x_i & & x_{i+1} & & \end{array}$$

## Part 2

1. Consider for  $x \in [0, 1]$  the p.d.e.  $\frac{\partial \phi}{\partial t} + \underline{\alpha \frac{\partial \phi}{\partial x}} = D \frac{\partial^2 \phi}{\partial x^2}$ , with  $D = \frac{1}{40}$ .

The boundary and initial conditions are  $\phi(0, t) = 0$ ,  $\phi(1, t) = 1$ ,  $\phi(x, 0) = x^2$ .

The equidistant spatial grid consists of 3 internal gridpoints and 2 boundary points.

For both  $\frac{\partial^2 \phi}{\partial x^2}$  and  $\frac{\partial \phi}{\partial x}$  2nd order central FDM discretisation is used.

- (a) 10 ~~1~~ Let  $\alpha = 0$  (so the underlined term is 'switched off').

What is the max. possible time step  $\delta t$  in case of the Explicit Euler method?

Hint: you may use any known stability limits directly.

~~2~~ Let  $\alpha = 1$  (so the underlined term becomes 'active'). Again Explicit Euler is used.

Show that the concept of positive coefficients leads to the same time step limit as for the case  $\alpha = 0$ . Hint: set up the discretisation first.

P.T.O.

- (b) 10 Let  $\alpha = 2$  (so the underlined term is again active), and use  $\delta t = \frac{1}{4}$ .

Determine the row of the linear system (i.e. all the relevant entries of the matrix and the RHS vector) that is related to  $x = \frac{1}{2}$ , when Implicit(!) Euler is used.

Hint: it helps to work in terms of the  $R$ -parameter.

- ~~X~~ 10 The p.d.e.  $\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$  is treated with 2nd order central spatial discretisation (FDM), on an equidistantly spaced grid, in combination with the Implicit(!) Euler method. Use Von Neumann stability analysis, to determine the (practical) stability limit (if any). Hint: use  $\phi_j^{(n)} = \rho^n e^{ij\varphi}$  in the discretisation and determine the amplification factor.

### Part 3

$$\begin{array}{ccc} R & (1-2R) & R \\ -R & (1+2R) & -R \end{array}$$

1. Consider for  $x \in [0, 1]$  the o.d.e.  $-2 \frac{d^2 u}{dx^2} + \frac{du}{dx} = 2x$ , with b.c.  $u(0) = \frac{du}{dx}(1) = 0$ .

- ~~X~~ 10 Using the Galerkin (weighted-residual) approach, determine the weak formulation of the problem and identify the bi-linear and linear forms.

- ~~X~~ 10 ~~X~~ Is the bi-linear form symmetric? Explain why.

~~X~~ Show that the bi-linear form is bounded.

- ~~X~~ 10 The domain will be covered with 3 grid points:  $x_0 = 0$ ,  $x_1 = \frac{1}{2}$ ,  $x_2 = 1$ .

We will use piecewise linear ('tent shape') basis functions, with  $\phi_i(x_j) = \delta_{ij}$ .

Use integration to evaluate  $A_{1,1}$  and  $A_{1,2}$ , which are part of the middle row of the linear system  $A\vec{c} = \vec{b}$ , where  $\vec{c} = [c_0, c_1, c_2]^T$ .

Total:

100

Write your name and student number on each page!

$$\begin{aligned} \Rightarrow \phi^{n+1} &= [\dots]^{-1} \phi^n \\ \left[ \dots \right] \phi^{n+1} &= \phi^n \left[ \dots \right]^{-1} \phi^0 \\ \int_0^1 u' v &= \underbrace{[u \cdot v]_0^1}_{=0} - \int_0^1 u v' dx \\ R &= \frac{D \delta t}{h^2} = \frac{1}{40} \frac{1}{4} \frac{8}{1} = \frac{1}{20} \\ \sqrt{\|\cdot\| + \left\| \frac{\partial}{\partial x} \right\|} \\ \|u\| &= \int_0^1 u u dx \\ \int_{x_i}^{x_{i+1}} \frac{d\phi_i}{dx} \phi_{i+1} dx &= \left( \frac{d\phi_i}{dx}, \phi_{i+1} \right)_{E_{i+1}} \\ &= \left( \frac{d\phi_i}{dx}, \phi_i \right)_{E_i} \end{aligned}$$